

Pressure and intermittency in passive vector turbulence

L. Ts. Adzhemyan,¹ N. V. Antonov,¹ A. Mazzino,^{2,3} P. Muratore-Ginanneschi⁴ and A. V. Runov¹

¹ *Department of Theoretical Physics, St Petersburg University, Uljanovskaja 1, St Petersburg, Petrodvorez, 198904 Russia*

² *CNR-ISIAtA, c/o Lecce University, 73100, Lecce, Italy*

³ *INFN-Department of Physics, Genova University, I-16146 Genova, Italy*

⁴ *Department of Mathematics, Helsinki University, P.O. Box 4, 00014, Helsinki, Finland*
(12 February, 2001)

We investigate the scaling properties a model of passive vector turbulence with pressure and in the presence of a large-scale anisotropy. The leading scaling exponents of the structure functions are proven to be anomalous. The anisotropic exponents are organized in hierarchical families growing without bound with the degree of anisotropy. Nonlocality produces poles in the inertial-range dynamics corresponding to the dimensional scaling solution. The increase with the Péclet number of hyperskewness and higher odd-dimensional ratios signals the persistence of anisotropy effects also in the inertial range.

PACS number(s): 47.10.+g, 05.10.Cc, 47.27.-i

In the last five years, much progress has been achieved in understanding of the physical origin of intermittency and anomalous scaling in fluid turbulence. The study of the passive advection of a scalar quantity (temperature, concentration of an impurity *etc*) by a random Gaussian field, white in time and self-similar in space, the rapid-change model [1], has played a crucial role in these developments. There, it was for the first time possible to demonstrate the emergence of anomalous scaling and to relate it to the existence of statistical conservation laws of the dynamics [2,3]. Such conservation laws appear as scaling zero modes of the hierarchy of linear operators governing the inertial-range dynamics of the correlation functions. The dominance of such zero modes has been confirmed in numerical experiments [4] and demonstrated analytically by means of the renormalization group and operator product expansion [5,6]. The importance of these results is seen from the fact that statistical invariants of dynamics are features of the passive advection by more general classes of turbulent velocity fields [7]. A review and more references can be found in Ref. [8].

The first-principle model of fluid turbulence is the Navier-Stokes equation, a nonlinear integro-differential equation for the solenoidal vector velocity field. Whether the relation between statistical invariants and anomalous scaling can be generalized to the nonlinear dynamics of the advecting velocity field remains an open and challenging issue. Setting aside the problem of the nonlinearity of interactions, the scope of the present Letter is to investigate how nonlocality affects scaling.

In the Navier-Stokes equation nonlocality arises through the pressure term fixed by the incompressibility condition for the velocity field. Here, we consider the passive advection of an incompressible vector field \mathbf{u} according to the most general dynamics consistent with the Galilean invariance:

$$\partial_t \mathbf{u} + (\mathbf{v} \cdot \partial) \mathbf{u} - \mathcal{A}(\mathbf{u} \cdot \partial) \mathbf{v} + \partial \mathcal{P} = \kappa \Delta \mathbf{u} + \mathbf{f}, \quad (1)$$

where \mathcal{P} is the pressure and κ is the diffusivity coefficient. The advection field \mathbf{v} is specified by the Kraichnan ensemble. It models a homogeneous and isotropic nonintermittent velocity field and, due to Gaussianity, is determined completely by the second-order structure function

$$S_{\alpha\beta}(\mathbf{r}) = \langle v_\alpha(\mathbf{x}, t) (v_\beta(\mathbf{x}, t') - v_\beta(\mathbf{y}, t')) \rangle = D \delta(t - t') |\mathbf{r}|^\xi [(d - 1 + \xi) \delta_{\alpha\beta} - \xi \hat{r}_\alpha \hat{r}_\beta] \quad (2)$$

with $\mathbf{r} \equiv \mathbf{x} - \mathbf{y}$ and $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$. Here $0 < \xi < 2$ is a kind of Hölder exponent which measures the “roughness” of the velocity field. In the renormalization group approach, it plays the same role as the parameter $\varepsilon = 4 - d$ in the theory of critical phenomena.

The vector field \mathbf{u} is stirred by an incompressible forcing \mathbf{f} varying only over large spatial scales. We assume the forcing \mathbf{f} to be delta-correlated in time and Gaussian in space with the covariance

$$\langle f_\alpha(\mathbf{x}) f_\beta(\mathbf{y}) \rangle = \delta(t - t') F_{\alpha\beta}(\mathbf{r}/L). \quad (3)$$

The parameter L is the integral scale of the stirring, and $F_{\alpha\beta}$ is a function which rapidly decays as $\mathbf{r}/L \rightarrow \infty$ and tends to a constant as $\mathbf{r}/L \rightarrow 0$.

As in the Navier-Stokes equation, the pressure \mathcal{P} is fixed by requiring Eq. (1) to be consistent with the incompressibility condition:

$$\Delta \mathcal{P}(\mathbf{x}, t) = (\mathcal{A} - 1) (\partial_\alpha v_\beta) (\partial_\beta u_\alpha). \quad (4)$$

The parameter \mathcal{A} modules the degree of nonlocality in the problem. The case $\mathcal{A} = 1$, known as the Kazantsev-Kraichnan kinematic dynamo model, represents the limit of local interactions. It describes the early stage of the growth of a magnetic seed in a turbulent fluid. However, the exponential growth of the magnetic field occurs only if ξ is large enough, in particular for $\xi > 1$ in three dimensions [9]. For smaller ξ , a steady state sets in exhibiting anomalous scaling behavior already for the pair correlation function. The corresponding exponents have been found exactly in the isotropic [9] and anisotropic [10–12] sectors.

As \mathcal{A} deviates from unity, the solution of the Poisson equation (4) introduces in the dynamics a nonlocal integral term. Cases of special interest are $\mathcal{A} = 0$ and $\mathcal{A} = -1$. For $\mathcal{A} = 0$ the stretching term $(\mathbf{u} \cdot \partial)\mathbf{v}$ vanishes and the dynamics becomes invariant under translations $\mathbf{u} \rightarrow \mathbf{u} + \text{const}$. The kinetic energy $\langle \mathbf{u}^2(\mathbf{x}, t) \rangle$ is conserved, as in the passive scalar and stochastic Navier-Stokes equation; see Ref. [13,14]. The case $\mathcal{A} = -1$ corresponds to the linearization of the Navier-Stokes equation around the rapid-change velocity field; in two dimensions it was studied in Ref. [15].

We are interested in the scaling properties of the correlation functions of the field \mathbf{u} in the inertial range, $L \gg |\mathbf{r}| \gg \ell \equiv (\kappa/D)^{1/\xi}$, where ℓ is the dissipative scale. The equal-time pair correlation function is governed by a linear *integro*-differential equation,

$$\partial_t C_{\alpha\beta} = \mathcal{D}_{\alpha\beta}^{\gamma\delta} C_{\gamma\delta}, \quad C_{\alpha\beta}(\mathbf{r}, t) = \langle u_\alpha(\mathbf{x}, t) u_\beta(\mathbf{y}, t) \rangle, \quad (5)$$

where the summation over repeated indices is implied. The explicit form of the inertial operator $\mathcal{D}_{\alpha\beta}^{\gamma\delta}$, of scaling dimension $-2 + \xi$, can be derived using functional integrations by parts [16] or the Dyson-Wyld equation [6,11]. It is not given here for the sake of brevity. Under the assumption that a steady state sets in, the inertial-range isotropic solution is sought in the time-independent scaling form:

$$C_{\alpha\beta}(\mathbf{r}) = C |\mathbf{r}|^{\zeta_2} [(d-1+\zeta_2)\delta_{\alpha\beta} - \zeta_2 \hat{r}_\alpha \hat{r}_\beta]. \quad (6)$$

After the adoption of the Ansatz (6), we define the integrals in Eq. (5) in the spirit dimensional regularization. Taking the trace then gives the desired equation for ζ_2 :

$$\mathcal{D}_{\alpha\alpha}^{\gamma\delta} C_{\gamma\delta} = D C |\mathbf{r}|^{-2+\xi+\zeta_2} \frac{(d-1)(\zeta_2 + \xi + d - 2)}{(\zeta_2 + \xi - 2)} T(\xi, \zeta_2, \mathcal{A}) = 0, \quad (7)$$

where

$$\begin{aligned} T(\zeta_2, \xi, \mathcal{A}) \equiv & -(\mathcal{A}-1)^2(d+1)\xi\zeta_2 + (\zeta_2 + \xi - 2) [d^2(\zeta_2 + \mathcal{A}^2\xi) + d(-\zeta_2 + \zeta_2^2 + \mathcal{A}^2\xi(\xi-1)) - (\zeta_2 - \mathcal{A}\xi)^2] - \\ & - 4(\mathcal{A}-1)(\mathcal{A}\xi - \mathcal{A} - d - 1) \frac{\Gamma(1+\xi/2)\Gamma(1+d/2+\zeta_2/2)\Gamma(2-\xi/2-\zeta_2/2)\Gamma(1+d/2)}{\Gamma(d/2+\xi/2+\zeta_2/2)\Gamma(-\zeta_2/2)\Gamma(2+d/2-\xi/2)}. \end{aligned} \quad (8)$$

Equation (7) has always a simple root for $\zeta_2 = 2 - d - \xi$ and for all $\mathcal{A} \neq 0, 1$ a simple pole for $\zeta_2 = 2 - \xi$. It is worth stressing that the pole arises because of the nonlocal terms in the inertial operator. The scaling $|\mathbf{r}|^{2-\xi}$ is marginal and produces logarithmic divergences at the edges of the inertial range which are reflected in the presence of the pole. In order to understand its physical meaning it is instructive to compare Eq. (7) with the passive scalar case [2,3]. The inertial operator governing the pair correlation function again has dimension $-2 + \xi$; its action onto a power function yields

$$\mathcal{D}|\mathbf{r}|^{-2+\xi+\zeta_2} = D C |\mathbf{r}|^{-2+\xi+\zeta_2} (d-1)\zeta_2(\zeta_2 + \xi + d - 2). \quad (9)$$

The root $\zeta_2 = 2 - d - \xi$ is again present. It corresponds to the dimensional solution of the equation with the right-hand side is proportional to $\delta(\mathbf{r})$ (or a constant in momentum space). The corresponding scaling behavior is admissible at scales much larger than the integral one [2]. The second root $\zeta_2 = 0$, a constant zero mode, is a consequence of energy conservation and as such it is also a root of Eq. (7) for $\mathcal{A} = 0$. Finally, setting $\zeta_2 = 2 - \xi$ in Eq. (9) solves the equation with a constant right-hand side (or $\delta(\mathbf{k})$ in the momentum space). The pole for $\zeta_2 = 2 - \xi$ in Eq. (8) thus corresponds to the scaling solution obtained by matching the forcing at large scales. Its cancellation for $\mathcal{A} = 0$ indicates the existence of a constant flux solution in the presence of kinetic energy conservation.

The transcendental equation $T(\zeta_2, \xi, \mathcal{A}) = 0$ has infinitely many solutions. The leading admissible exponent behaves as $\zeta_2^+ = O(\xi)$ for small ξ . It can be obtained from Eq. (8) within perturbation theories in $1/d$ or ξ :

$$\begin{aligned}\zeta_2^+ &= -\mathcal{A}^2\xi + \frac{\mathcal{A}^2\xi}{d} \left\{ \frac{(\mathcal{A}-1)\Gamma(1+\xi/2)\Gamma(1+(\mathcal{A}^2-1)\xi/2)}{\Gamma(1+\mathcal{A}^2\xi/2)} - \frac{\xi(\mathcal{A}+1)(\mathcal{A}^3\xi - \mathcal{A}^2\xi + \mathcal{A}\xi + \mathcal{A} - \xi + 1)}{(\mathcal{A}^2\xi - \xi + 2)} \right\} + O(1/d^2) \\ &= -\xi \frac{\mathcal{A}^2(d-1)(d+2)}{(d^2 + \mathcal{A}^2 + \mathcal{A}d - 3)} - \frac{\xi^2 \mathcal{A}^2(d-1)}{2d(d^2 + \mathcal{A}^2 + \mathcal{A}d - 3)^2} \left\{ d^3(\mathcal{A}+1)^2 + (d^2 - 2d + 4)(3\mathcal{A}^2 + 2\mathcal{A} + 3) \right\} + O(\xi^3). \quad (10)\end{aligned}$$

The nonlocality parameter \mathcal{A} enters the perturbative expansions from the first order. The fact reflects the balance between local and nonlocal contributions in the zero-mode equation. It is worth noting that up to first order in $1/d$ the exponent of the linearized Navier-Stokes and of the magnetic, local, model coincide.

The next-to-leading correction exponent has the form $\zeta_2^- = 2 - \xi + \mathcal{A}(\mathcal{A}-1)(d+2)\xi/(d^2 + \mathcal{A}^2 + \mathcal{A}d - 3) + O(\xi^2)$.

Nonperturbative solutions of Eq. (7) can be obtained analytically only in the case $\mathcal{A} = 0$, where $\zeta_2^+ = 0$ and $\zeta_2^- = 2 - \xi$ exactly, and in the local case $\mathcal{A} = 1$ when it reduces to a third-order algebraic equation. Besides the subleading solution $\zeta_2^- = 2 - \xi$, the result of Ref. [9] is recovered. In general, the behaviour of the exponents ζ_2^\pm depends qualitatively on the value of \mathcal{A} . Below we focus our attention only on the behaviors of the leading exponent ζ_2^+ vs ξ for $d = 3$ and in the physically interesting case $\mathcal{A} \in [-1, 1]$. More detailed analysis will be reported elsewhere.

When $-0.581 < \mathcal{A} < 0.613$, the solution ζ_2^+ exists and is strictly negative within the entire interval $0 \leq \xi \leq 2$, except for the case $\mathcal{A} = 0$ when ζ_2^+ vanishes identically. For the other values of \mathcal{A} , the real solution exists only in the subinterval $0 \leq \xi \leq \xi_c$, ($\xi_c(\mathcal{A}, d) \leq 2$ is some critical value) where it is again strictly negative. For $\xi \geq \xi_c$, the solution ζ_2^+ coalesces with the closest unphysical branch, $\zeta_2 = -d + O(\xi)$, and they both become complex: the effect known for the magnetic model [9], where $\xi_c(1, 3) = 1$. It was argued in Refs. [9,10] that this critical value of ξ coincides with the threshold for the dynamo effect (exponential growth of the pair correlation function).

It is worth noticing that in the small subinterval, $-0.613 < \mathcal{A} < -0.581$, both solutions again become real near to $\xi = 2$ (see Fig. 1).

Coefficients of the expansions in ξ for ζ_2^\pm diverge at $(d^2 + \mathcal{A}^2 + \mathcal{A}d - 3) = 0$ (this can happen only for $d \leq 2$, and for $d = 2$ only for $\mathcal{A} = -1$). Nonperturbative analysis shows that in the region $(d^2 + \mathcal{A}^2 + \mathcal{A}d - 3) < 0$, the effective diffusivity coefficient is negative at large scales, which makes the system unstable with respect to any small perturbation. In the limit $(d^2 + \mathcal{A}^2 + \mathcal{A}d - 3) \rightarrow 0$, the critical value ξ_c vanishes, and the solution for the leading inertial-range exponent disappears. We thus conclude that the solution $\zeta_2 = -\xi$, reported in Ref. [15] for $d = 2$ and $\mathcal{A} = -1$, in fact corresponds to the large-scale exponent $\zeta_2 = 2 - d - \xi$ mentioned above and not to the inertial range.

Nonperturbative analysis can be repeated for the anisotropic sectors of the pair correlation function; see Refs. [10–12] for $\mathcal{A} = 1$ and [13,14] for $\mathcal{A} = 0$. Anisotropic contributions can be excited, e.g., in the presence of a constant background field [10,11]. The correlation can be expanded in components of definite total angular momentum j with radial dependence of the powerlike form (parity considerations allow only even j in this expansion):

$$\mathcal{C}_{\alpha\beta}(\mathbf{r}) \simeq \sum_{j=0}^{\infty} C_{\alpha\beta}(j) |\mathbf{r}|^{\zeta_2(j)} P_j(\hat{\mathbf{r}}). \quad (11)$$

However, up to first order in ξ these exponents can be most conveniently calculated using the renormalization group and operator product expansion. We refer the reader to the literature [5,6,11,17] for details and present only the result:

$$\begin{aligned}\zeta_2^{(j)} &= (j-2) + \frac{\xi(d+2)}{(\mathcal{A}^2 + d^2 + \mathcal{A}d - 3)(2j+d-6)(2j+d-4)(2j+d-2)} \\ &\times \left\{ (j^4 + 4j^3d + 2j^2d^2 - 10j^3 - 22j^2d - 6jd^2 + 35j^2 + 38jd + 6d^2 - 50j - 24d + 24)\mathcal{A}^2 \right. \\ &\quad \left. + (j-2)(j^2 + 2jd + d^2 - 5j - 4d + 4)[2\mathcal{A}(j+d-3) + (j-3)] \right\}, \quad (12)\end{aligned}$$

where $j \geq 2$. The dependence on the angular momentum indicates the existence of an hierarchy of exponents growing with j without an upper bound. The result is in agreement with previous findings for the passive advection of the scalar [17] and magnetic [10–12] fields and with the nonlocal case $\mathcal{A} = 0$ [13,14].

The renormalization group and operator product analysis is also extended to correlation functions of arbitrary order. The key role is played by the dimensions $\Delta[n, j]$ associated with the j th rank tensor composite operators $u_{\alpha_1} \dots u_{\alpha_j} (\mathbf{u}^2)^l$, where $n \equiv 2l + j$. Let $\mathcal{D}_{n,p}(\mathbf{x}, \mathbf{y})$ be some equal-time scalar correlation function of the n th and p th powers of the field \mathbf{u} . For the leading term in the j th anisotropic sector we obtain:

$$\mathcal{D}_{n,p}(\mathbf{x}, \mathbf{y}) \propto (|\mathbf{r}|/\ell)^{-\Delta_{n,jn} - \Delta_{n,jp}} (|\mathbf{r}|/L)^{\Delta_{n,j}} P_j(\hat{\mathbf{r}}), \quad (13)$$

where $j_k = 0$ or 1 for even or odd k , respectively; cf. [6,17] for the scalar and [11] for the magnetic cases. Up to order $O(\xi)$, we have obtained

$$\Delta[n, j] = \frac{n\xi}{2} + \frac{\xi \mathcal{A}^2 [2n(n-1) - (d+1)(n-j)(d+n+j-2)]}{2(d^2 + \mathcal{A}^2 + \mathcal{A}d - 3)} + O(\xi^2). \quad (14)$$

This result, in particular, demonstrates the persistence of anisotropy at small scales. Like in the scalar case [18], dimensionless ratios of the form $R_{2n+1} \equiv \mathcal{D}_{2n+1,0}/\mathcal{D}_{2,0}^{(2n+1)/2}$ can be constructed using Eqs. (13) and (14) and evaluated at the dissipative scale ℓ . In terms of the Péclet number $\text{Pe} \propto (L/\ell)^{1/\xi}$ we obtain

$$\mathcal{R}_{2k+1} \propto \text{Pe}^{\sigma_{2k+1}}, \quad \sigma_{2k+1} = \frac{\mathcal{A}^2 (d-1) (4k^2 - d - 2)}{2(d^2 + \mathcal{A}^2 + \mathcal{A}d - 3)} + O(\xi). \quad (15)$$

The ratio for $k = 2$ (the so-called hyperskewness factor) diverges as $\text{Pe} \rightarrow \infty$. This means that the conclusions drawn in the scalar [17] and magnetic [11,19] cases regarding the persistence of anisotropy at small scales remain valid in the presence of the pressure term.

To summarize, the study of the pressure effects on the inertial-range scaling behavior of a passive vector quantity advected by the rapid change model reveals a double breakdown of the formal dimensional analysis. On one hand, the leading scaling exponents are anomalous and governed by statistical conservation laws (zero modes) of the inertial-range dynamics: a phenomenon already known for a local vector (magnetic) model. On the other, as a peculiar feature of nonlocality, the dimensionally regularized inertial operator develops a pole corresponding to the scaling solution obtained by formally matching the forcing at large scales. When the pressure term is present, nonanomalous scaling of the pair structure function with the dimensional exponent $2 - \xi$ is realized only in the case when the stretching term in Eq. (1) is absent and the energy conservation enforces a constant flux solution.

The authors gratefully acknowledge valuable discussions with I. Arad, A. Celani, A. Kupiainen and M. Vergassola.

The work of L. Ts. A., N. V. A. and A. V. R. was supported in part by the RFFI Grant No. 99-02-16783 and GRACENAS Grant No. E00-3-24. A. M. was partially supported by the INFM PA project No. GEPAIGG01. P. M. G. was partially supported by EC Grant FMRX-CT98-0175. P. M. G. thanks the “Centro Internacional de Ciencias” of Cuernavaca and the organizers of the workshop “Scaling and universality in strongly nonlinear systems” during which a part of this work has been performed.

-
- [1] R. H. Kraichnan, Phys. Fluids **11**, 945 (1968); Phys. Rev. Lett. **72**, 1016 (1994); *ibid.* **78**, 4922 (1997).
 - [2] M. Chertkov, G. Falkovich, I. Kolokolov and V. Lebedev, Phys. Rev. E **52**, 4924 (1995); M. Chertkov and G. Falkovich, Phys. Rev. Lett. **76**, 2706 (1996).
 - [3] K. Gawędzki and A. Kupiainen, Phys. Rev. Lett. **75**, 3834 (1995); D. Bernard, K. Gawędzki and A. Kupiainen, Phys. Rev. E **54**, 2564 (1996).
 - [4] U. Frisch, A. Mazzino and M. Vergassola, Phys. Rev. Lett. **80**, 5532 (1998).
 - [5] L. Ts. Adzhemyan, N. V. Antonov and A. N. Vasil’ev, Phys. Rev. E **58**, 1823 (1998); L. Ts. Adzhemyan, N. V. Antonov, V. A. Barinov, Yu. S. Kabrits and A. N. Vasil’ev, Phys. Rev. E **63**, 025303(R) (2001).
 - [6] L. Ts. Adzhemyan and N. V. Antonov, Phys. Rev. E **58**, 7381 (1998); L. Ts. Adzhemyan, N. V. Antonov, M. Hnatich and S. V. Novikov, Phys. Rev. E **63**, 016309 (2001).
 - [7] A. Celani and M. Vergassola, Phys. Rev. Lett. **86**, 424 (2001).
 - [8] B. I. Shraiman and E. Siggia, Nature **405**, 639 (2000).
 - [9] M. Vergassola, Phys. Rev. E **53**, R3021 (1996).
 - [10] A. Lanotte and A. Mazzino, Phys. Rev. E **60**, R3483 (1999).
 - [11] N. V. Antonov, A. Lanotte and A. Mazzino, Phys. Rev. E **61**, 6586 (2000).
 - [12] I. Arad, L. Biferale and I. Procaccia, Phys. Rev. E **61**, 2654 (2000).
 - [13] L. Ts. Adzhemyan and A. V. Runov, Vestnik SPBU, Ser. Phys. Chem., issue 1 (No 4) (2001); L. Ts. Adzhemyan, N. V. Antonov and A. V. Runov, *A model of the passively advected vector field: Anomalous scaling, nonlocality and anisotropy* (to be published).
 - [14] I. Arad and I. Procaccia, nlin.CD/0011026.
 - [15] K. Yoshida and Y. Kaneda, Phys. Rev. E **63**, 016308 (2001).
 - [16] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).

- [17] N. V. Antonov, Phys. Rev. E **60**, 6691 (1999), Physica D **144**, 370 (2000); *nlín.CD/0007015*; N. V. Antonov and J. Honkonen, Phys. Rev. E **63** (2001).
- [18] A. Pumir, Europhys. Lett. **34**, 25 (1996); Phys. Rev. E **57**, 2914 (1998).
- [19] N. V. Antonov, J. Honkonen, A. Mazzino and P. Muratore-Ginanneschi, Phys. Rev. E **62**, R5891 (2000).

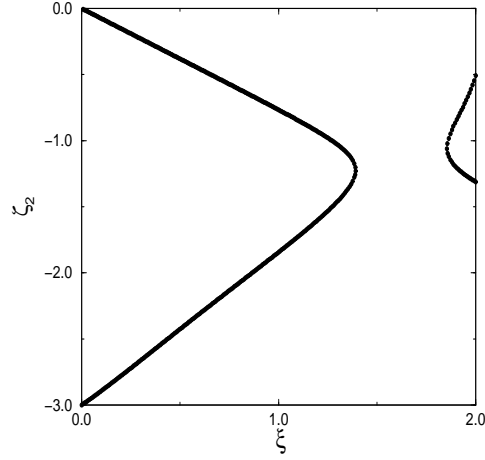


FIG. 1. The leading exponent ζ_2^+ (upper branches) and the unphysical exponent $-d + O(\xi)$ (lower branches) *vs* ξ for $d = 3$ and $\mathcal{A} = -0.610$, within the small interval $\mathcal{A} \in [-0.613, -0.581]$ (see text). The coalescence occurs at $\xi_c = 1.390$ after which solutions become complex. The two branches becomes again real for $\xi > 1.850$.